On the D-Width of Directed Graphs

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1 The Strong Visible Robber Game: Basic Properties

The *cops and strong visible robber game*, defined in [3], is played according to the following rules:

Initially, the cops occupy some set of $X \subseteq V$ vertices, with $|X| \leq k$, and the robber is placed on some vertex $v \in V - X$. At any time, some of the cops can reside outside the graph, say, in a helicopter. In each round, the cop player chooses the next location $X' \subseteq V$ for the cops. The stationary cops in $X \cap X'$ remain in their positions, while the others go to the helicopter and fly to their new position. During this, the robber player, knowing the cops' next position X' from wiretapping the police radio, can run at great speed to any new position v', provided there is both a (possibly empty) directed path from v to v', and a (possibly empty) directed path back from v' to v in $G - (X \cap X')$, that is, he has to avoid to run into a stationary cop, and to run along a path in and to stay in the same strongly connected component of the remaining graph induced by the non-blocked vertices. Afterwards, the helicopter lands the cops at their new positions, and the next round starts, with X' and v' taking over the roles of X and v, respectively.

The cop player wins the game if eventually a cop lands on the robber's current position or robber cannot move any more, and the robber player wins if the robber can escape indefinitely.

Instead of letting the robber choose a single vertex in each turn, we prefer to think of the robber's positions as strongly connected components of G, and of the robber choosing in each turn a strongly connected component R' in $G - X'$ such that R and R' are in the same strongly connected subset of $G - X \cap X'$.

Formally, we can define a *play* of the above game as a finite or infinite word over the set of *game positions* $\pi = (X_0, R_0)(X_1, R_1)\ldots$ with $X_0 = \emptyset$, $R_0 = V$, and for each $i \geq 1$ holds $X_i, R_i \subseteq V$, and R_{i+1} is a strongly connected component of $G - X_{i+1}$ such that R_i and R_{i+1} are in the same strongly connected component of $G - (X \cap X')$. Furthermore, if π is finite, then the last game position in π is $(X, \emptyset)$ for some $X \subseteq V$. For reasoning about suffixes of plays, it is convenient to allow arbitrary starting positions (X_0, R_0) in the definition above. Then we say that π is *a play from position* (X_0, R_0). The cop player wins the play π if it is finite and ends with position $(X, \emptyset)$ for some $X \subseteq V$. The robber player wins the play if it is infinite and contains no positions of the form $(X, \emptyset)$ with $X \subseteq V$.

A *strategy* σ for k cops is a function that, for each given game position (X, R), returns the next position X' for the cops, and we require that each position holds $|\sigma(X, R)| \leq k$. A game position (X, R) is *reachable* (w.r.t. strategy σ) if there is a play consistent with σ in which (X, R) occurs. A play π is *consistent* with a strategy σ if for each $i \geq 0$ holds $X_{i+1} = \sigma(X_i, R_i)$. We say that σ is a k-cop *winning strategy* if all plays consistent with σ are winning for the cops. A strategy σ is *robber-monotone* if for all plays consistent with σ holds $R_{i+1} \subseteq R_i$, i.e. the set of robber positions is a monotonically decreasing sequence, and σ is *cop-monotone* if the cops can never revisit a vertex once it has been vacated, that is, for all $i \leq j \leq k$ holds $X_i \cap X_k \subseteq X_j$ in all plays consistent with σ.

It is known that every cop-monotone strategy is robber-monotone, but that the converse is not true in general [3]. Still, any strategy for k cops can be simulated by a winning strategy that uses only $O(k^2)$ cops [2].

Lemma 1. *Assume G is a partial directed union of two graphs G_1 and G_2. Then $dw(G) = \max\{dw(G_1), dw(G_2)\}$.*

Proof. We have $dw(G) \geq dw(G[U])$ for every subgraph induced by a subset U of the vertex set of G. Since G_1 and G_2 are induced subgraphs of G, $dw(G) \leq \max\{dw(G_1), dw(G_2)\}$.

For the other direction, assume the $k+1$ cops have winning strategies π_1 and π_2 for both G_1 and G_2. Then they can combine these to form a winning strategy π for G as follows: They place no cop at all on the graph in the first move, and wait whether the robber's initial position is on G_1 or G_2; the robber cannot choose a strongly connected subset of $V(G)$ that intersects with both subgraphs, as this would contradict the fact that G is a partial directed union of G_1 and G_2. If the robber chooses a vertex in G_1, he cannot enter a vertex in G_2 since he always to remain in the same strongly connected component of G, which is either entirely in G_1 or entirely in G_2. So the cops can from now on play according to the strategy π_1. A symmetric argument applies if the robber's initial position is a vertex in G_2. □

2 A separator theorem for D-width

The notion of D-width was introduced in [5] as an attempt to a more natural generalization of (undirected) treewidth to directed graphs than the one originally proposed in [3]. Symmetric graphs of small undirected treewidth enjoy the property that they admit small separators [4]. This allows for efficient divide-and-conquer algorithms for many problems on graphs of bounded treewidth that are intractable (e.g. **NP**-complete) for general graphs, see e.g. [1]. The goal of this section is to show that digraphs of bounded D-width have small weak separators, the latter being a generalization of separators to directed graphs.

Definition 2. *A* D-Decomposition *of a directed graph G is an undirected tree T with $V(T) = I$, together with a family of* bags $\{X_i | i \in I\}$, *where every bag X_i is a subset of $V(G)$, with the property that for every strong subset S in G holds: (D1)*

The set of vertices whose bags X_i intersect with S, formally $\{ i \mid X_i \cap S \neq \emptyset \}$ induce a (connected) subtree T_S of G. (D2) S intersects with at least one X_i such that X_i is the bag of some vertex i in $V(T)$. The width *of the D-decomposition is the minimum w such that each bag contains at most $w+1$ digraph vertices.*

Remark. We note that the original definition of D-width from [5] is a bit more strict. With notation as above, the original condition (D1) imposes the condition that the subgraph induced by the *edge set* to $\{\{i,j\} \in E(T) \mid X_i \cap X_j \cap S \neq \emptyset\}$ remains connected. But the strong correspondence of D-decompositions and the strategy DAGs for cop-monotone strategies established in [2] does in fact only hold for the definition used here. (The problematic situation occurs when in the cops and robber game two consecutive locations X and X' of the cop player are disjoint. In this case the additional restriction gives a disconnected subgraph for the more restricted definition.)

Suppose $(T, \{X_i | i \in I\})$ is a D-Decomposition for G.

Lemma 3. *For strong subsets S,S' of G with $S' \subseteq S$, let T_S and $T_{S'}$ the subtrees of T defined in (D1). Then $T_{S'}$ is a subtree of T_S.*

Proof. By definition, the bag X_i of every tree vertex i in $T_{S'}$ intersects with S' and hence also with S. Thus i is also a vertex of the subtree T_S. □

For each vertex i in I of the tree T, the connected components of $T - i$ are called the *branches at* i. The branches at i are in natural correspondence with the edges of T incident with i.

Lemma 4. *For $i \in I$ and $v \in V(G)$, either v is in the bag X_i of i, or there is a unique branch of T at i which contains all j in I with v in X_j.*

If $v \notin X_i$, the existence of at least one such branch follows from (D2), and its uniqueness from (D1). For $v \notin X_i$, let $T_i(v)$ denote this branch.

Lemma 5. *If there is a strong subset S in $V(G)$ with $X_i \cap S = \emptyset$, then there is a unique branch $T_i(S)$ such that for all $s \in S$ holds $T_i(s) = T_i(S)$.*

Proof. Assuming S is a strong subset of $V(G)$ that does not intersect with X_i, it suffices to show that for each pair of vertices v, w in S holds $T_i(v) = T_i(w)$: This gives the desired unique branch $T_i(S) = T_i(v)$.

By (D1), the bags intersecting with S form induce a subtree T_S of T, and by (D2), this subtree is nonempty. Since X_i does not intersect with S, i is not a vertex of T_S, but by Lemma 3, all vertices whose bags contain v or v' are in T_S. In other words, T_S forms a subtree of $T - i$ and both v and v' are in the same branch at i.

Definition 6. *For a digraph G, let $X \subseteq V(G)$. Two vertices $v, v' \in V(G) \setminus X$ are* weakly separated *in G by a set of vertices $X \subseteq V(G)$ if every strong subset S in G containing both v and v' intersects with X. In a similar manner, two sets of vertices $Y, Z \subseteq V(G) \setminus X$ of vertices in G are weakly separated by X if every strong subset S in G intersecting with both Y and Z intersects with X.*

One would expect from any definition of weak separators that if Y and Z are separated by X, then Y and Z are disjoint. This is indeed the case: Assume $v \in Y \cap Z$. Then $\{v\}$ is a strong subset not intersecting with Y and Z, but not with X, contradiction.

Let B be a branch at vertex i. Define the *crop load* of B, as the set of digraph vertices $v \in V(G) \setminus X_i$ such that $T_i(v) = B$. By Lemma 5, being in the crop load of the same branch at i induces an equivalence relation on $V(G) \setminus X_i$.

Lemma 7. *Let B and B' be two distinct branches at i. Then the crop loads of B and B' are pairwise disjoint. Furthermore, the crop load of each branch at i equals the union of zero or more strong components of $G - X_i$.*

Proof. The counterpositive of Lemma 5 in terms of weak separators states that if $v, v' \notin X_i$ and $T_i(v) \neq T_i(v')$ then v and v' are weakly separated in G by X_i. Thus if v is in the crop load of B and v' is in the crop load of B', then $v \neq v'$, and they belong to different strongly connected components of $G \setminus X_i$. Next, assume $T_i(v) = B$ and another digraph vertex v' belongs to the same strong component C in $G \setminus X_i$. Since C is a strong subset in G not intersecting with X_i, again by Lemma 5, this implies that $T_i(v') = B$. □

Lemma 8. *Let $e = \{j, j'\}$ be a tree edge of T, and let N, N' be the tree vertex sets of the two components of $T - e$. Let X denote the set of digraph vertices $(X_j \cap X_{j'})$. Then $X_j \cap X_{j'}$ weakly separates the digraph vertex sets $\bigcup_{i \in N} X_i \setminus (X_j \cap X_{j'})$ and $\bigcup_{i \in N'} X_i \setminus (X_j \cap X_{j'})$ in G.*

Proof. We onsider two cases: in the first case $X_j \subseteq X_{j'}$ or $X_{j'} \subseteq X_j$. Then Lemma 7 applies for the branches at j or for the branches at j', and we are finished.

In the case $X_j \neq X_{j'}$, let $Y = \bigcup_{i \in N} X_i \setminus (X_j \cap X_{j'})$ and $Z = \bigcup_{i \in N'} X_i \setminus (X_j \cap X_{j'})$. We root the D-Decomposition at j and play the monotone cops and robbers game with X_j being the cops' first position according to the monotone winning strategy that corresponds to the D-Decomposition rooted at j (see [2] for details). We place the robber at the initial vertex v in G such that the cops' response according to the strategy is to place the cops at $X_{j'}$. Then the robber sees the cops approaching their next position $X_{j'}$ in their helicopters, and has to choose to reside in a strong subset of $V(G) \setminus X = Y \cup Z$ in the non-blocked induced subgraph of G. Since the cops play according to a monotone strategy, placing the robber at a vertex in $X_j \setminus X_{j'}$ will let the robber win. However, the cops' strategy is winning, and thus the move must be forbidden for the robber because every strong subset of $Y \cup Z$ has an empty intersection with X_j. A symmetric argument applies if we root the D-Decomposition at $X_{j'}$, hence no strong subset of $V(G) \setminus X$ intersects with $X_j \cup X_{j'}$. Since also every singleton subset of the former set is a strong subset of $V(G) \setminus X = Y \cup Z$ has empty intersection with $X_j \cup X_{j'}$, $(Y \cup Z) \cap (X_j \cup X_{j'}) = \emptyset$.

In particular, this implies that the sets Y and Z are subsets of the crop loads of two branches B and B' at j. But $V(B) \subseteq N$ and $V(B') \subseteq N'$, and N and

N' partition the set of tree vertices I, thus the two branches are distinct. By Lemma 7, this implies that $Y \cap Z = \emptyset$.

For the sake of contradiction, assume now there is a strong subset S in $G - X$ containing both some $y \in Y$ and some $z \in Z$ which does not intersect with $X = X_j \cap X_{j'}$. As explained above, no strong subset of $V(G) \setminus X$ intersects with $X_j \cup X_{j'}$. Hence $S \subseteq Y \cup Z$, and since Y and Z are disjoint, the assumed set S containing both x and y cannot exist. □

Theorem 9 (Directed Separator Theorem). *Let G be a digraph and assume $(T, \{X_i | i \in I\})$ is a D-Decomposition of width at most $w+1$ for G. Then for each $Q \subseteq V(G)$, there exists a set of digraph vertices X with $|X| \leq w+1$ such that every strong component of $G - X$ has at most $\frac{1}{2}|Q \setminus X|$ vertices that are in Q.*

Proof. We distinguish two cases.

Case 1. Some $i \in I$ has the property that for each branch B of T at i,

$$|\{\, q \in Q \setminus X_i \mid T_i(q) = B \,\}| \leq \frac{1}{2}|Q \setminus X_i|.$$

In this case, the bag X_i is a valid choice for X that satisfies the statement of the theorem: We have $|X_i| \leq w+1$ and by Lemma 5, each pair of digraph vertices v, v' such that $T_i(v) \neq T_i(v')$ is weakly separated by X_i.

Case 2. For each $i \in I$ there is a branch B_i of T at i, such that

$$|\{\, q \in Q \setminus X_i \mid T_i(q) = B_i \,\}| > \frac{1}{2}|Q \setminus X_i|.$$

Intuitively, in this case, for each bag X_i there is a branch at i that is "too heavy" for X_i to be a weak separator with the required properties. Now the idea is to use a set $X_j \cap X_j'$ as weak separator, as provided by Lemma 8. We do this in a way such that if there is still a too heavy branch, then it is already almost a good branch, and we can remedy this by adding vertices from, say X_j, to the weak separator.

For $i \in I$, let e_i be the edge of T incident with i and with some vertex of B_i. Since T has fewer edges than vertices, by the pigeon-hole principle, there exist distinct vertices $j, j' \in I$ such that $e_j = e_{j'}$. Then $\{j, j'\}$ is an edge in T, j' belongs to the branch B_j, and j belongs to the branch $B_{j'}$. We now prepare to apply Lemma 8: Let N and N' be the vertex sets of $B_{j'}$ and B_j, respectively. Then N and N' are the vertex sets of the two components of the forest $T - \{j, j'\}$. Let $Y = \bigcup_{i \in N} X_i \setminus (X_j \cap X_{j'})$ and $Z = \bigcup_{i \in N'} X_i \setminus (X_j \cap X_{j'})$. The set $X_j \cap X_{j'}$ might be a candidate for a separator that satisfies the required conditions on the size of the strong components. Let's see.

Without loss of generality, we assume that Y has more elements in Q than Z, that is

$$|Y \cap Q| \geq |Z \cap Q|. \tag{1}$$

Recall that we assumed above that the branch B_j is too heavy for X_j being a separator, that is $|\{\, q \in Q - X_j \mid T_j(q) = B_j \,\}| > \frac{1}{2}|Q \setminus X_j|$. Since N' is the vertex set of the branch B_j and $\{\, q \in Q - X_j \mid T_j(q) = B_j \,\} = Z \cap Q$, we have

$$|Z \cap Q| > \frac{1}{2}|Q \setminus X_j|. \tag{2}$$

By Lemmata 7 and 8, Y and Z partition the vertex set of $G - (X_j \cap X_{j'})$. Since $X_j \subseteq Y$, we have

$$Q \setminus X_j = (Z \cap Q) \cup ((Y \setminus X_j) \cap Q),$$

and the union on the right-hand side is disjoint. Putting this into Inequality (2), we get

$$|Q \setminus X_j| = \underbrace{|Z \cap Q|}_{> \frac{1}{2}|Q \setminus X_j|} + |(Y \setminus X_j) \cap Q|$$

and thus

$$|(Y \setminus X_j) \cap Q| < |Z \cap Q|.$$

By Equations (1) and (2), we can draw a number of elements in X_j out of Y such that Equation (1) becomes equality: There exists $U \subseteq X_j \setminus (X_j \cap X_{j'})$ such that

$$|(Y \setminus U) \cap Q| = |Z \cap Q|. \tag{3}$$

Since Y and Z partition the set of digraph vertices in $G - (X_j \cap X_{j'})$, the two sets $Y \setminus U$ and Z partition the set $V(G) \setminus ((X_j \cap X_{j'}) \cup U)$. Intersecting both former sets with Q, we get a partition of $Q \setminus ((X_j \cap X_{j'}) \cup U)$, by Equation (3), this partition is exactly balanced:

$$|(Y \setminus U) \cap Q| = |Z \cap Q| = \frac{1}{2}|Q \setminus ((X_j \cap X_{j'}) \cup U)| \tag{4}$$

Now let $X = (X_j \cap X_{j'}) \cup U$. Since X contains the set $(X_j \cap X_{j'})$, Lemma 8 tells us that X weakly separates Y and Z and hence also every pair of respective subsets. Since every strong subset in $G - X$ is either a subset of $Y \setminus U$ or of Z, Equation (4) tells us that X also satisfies the balancing condition in the statement of the theorem. Finally, since $U \subseteq X_j$, we have $X \subseteq X_j$ and therefore $|X| \leq w + 1$ holds. □

Lemma 10 (Clique Containment Lemma). *If G has a complete symmetric subgraph K_r of cardinality r, then each D-decomposition for G has a bag containing all vertices of K_r.*

Proof. Clear by the alternative characterization by strategy DAGs of cops and robbers games: Assume in every reachable game position there is one vertex in K_r that is not occupied by the cops. This gives immediately an escape strategy for the robber on K_r, and hence on G.

Lemma 11 (Biclique Containment Lemma). *If G has a symmetric biclique $K_{r,s}$ as subgraph, $K_{r,s} = (A \uplus B, A \times B \cup B \times A)$, with $|A| = r$ and $|B| = s$, then then each tree decomposition for G has a bag containing all vertices of A or a bag containing all vertices of B.*

Proof. Similar to above: If in each vertex of the corresponding strategy dag, there is a free position in set A and a free position in set B, then the robber player can alternately move between the sets A and B and escape indefinitely.

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